Double Bubbles in the Three-Torus

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CONTENTS

We present a conjecture, based on computational results, on the area-minimizing method of enclosing and separating two arbitrary volumes in the flat cubic three-torus, $\mathbf{T}^3$. For comparable small volumes, we prove that the standard double bubble from $\mathbf{R}^3$ is area-minimizing.

1. INTRODUCTION

The classical isoperimetric problem seeks the least-area way to enclose a single region of prescribed volume. About 200 BC, Zenodorus argued that a circle is the least-perimeter enclosure of prescribed area in the plane (see [Heath 60]). In 1884, Schwarz [Schwarz 1884] proved by symmetrization that a sphere minimizes surface area for a given volume in $\mathbf{R}^3$. Isoperimetric problems arise naturally in many parts of modern mathematics; Ros provides a nice survey [Ros 01].

The double bubble problem is a two-volume generalization of the classical isoperimetric problem. In 2000, Hutchings, Morgan, Ritoré, and Ros proved that the standard double bubble was the unique least-area way to enclose and separate two volumes in $\mathbf{R}^3$ ([Hutchings et al. 00], [Hutchins et al. 02], [Morgan 00, Chapter 14]). In this paper, we present a conjecture, based on substantial numerical evidence, on the least-area way to enclose and separate two volumes in the three-torus.

Conjecture 1.1. (Central Conjecture 2.1.) *The ten different types of two-volume enclosures pictured in Figure 1 comprise the complete set of surface area-minimizing double bubbles for the flat cubic three-torus* $\mathbf{T}^3$.

The numerical results summarized in Figure 2 indicate the volumes for which we conjecture each type of double bubble minimizes surface area.

In Section 4, we show that for two small comparable volumes, the standard double bubble from $\mathbf{R}^3$ is optimal in $\mathbf{T}^3$. Specifically, we prove the following theorem:

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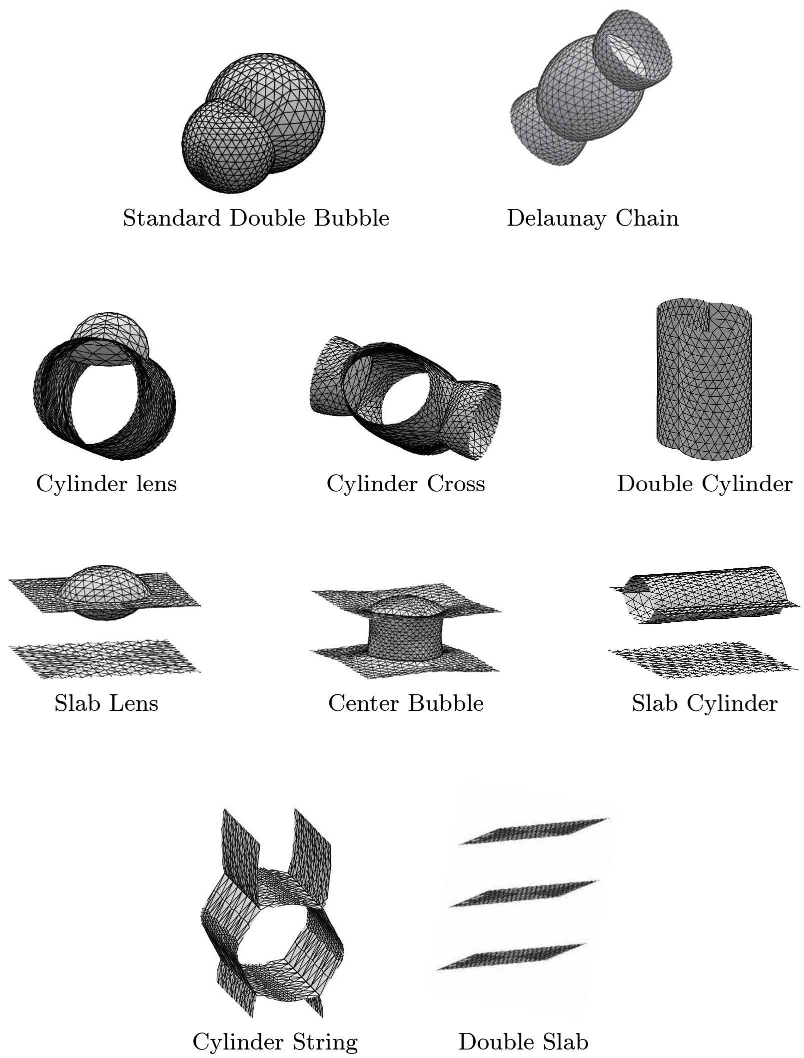

FIGURE 1. Catalog of conjectured minimizers.

Theorem 1.2. *Let M be a flat Riemannian manifold of dimension three or four such that M has compact quotient by its isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$, such that if $0 < v < \epsilon$, an area-minimizing double bubble in M of volumes $v, \lambda v$ is standard.*

We conclude the paper with some extensions to the primary conjecture that address noncubic tori and additional volume constraints.

1.1 Existence and Regularity

Using the language of geometric measure theory, bubble clusters can be expressed as rectifiable currents, varifolds, or $(\mathbf{M}, \epsilon, \delta)$-minimal sets [Morgan 00]. In three dimensions, area-minimizing bubble clusters exist and consist of constant-mean-curvature surfaces meeting smoothly in threes at 120° along smooth curves, which meet in fours at a fixed angle of approximately 109° ([Taylor 76, Theorems II.4, IV.5, IV.8], or [Morgan 00, Section 13.9]).

1.2 Recent Results

In 2000, Hutchings, Morgan, Ritoré, and Ros announced a proof that the standard double bubble, the familiar shape consisting of three spherical caps meeting one another at 120-degree angles, provides the area-minimizing method of enclosing and separating two volumes in $\mathbf{R}^3$ ([Hutchins et al. 02], [Hutchings et al. 00]). The key features of the proof are a component bound developed by Hutchings [Hutchings 97, Theorem 4.2] and an instability argument. Reichardt et al. [Reichardt et al. 03] extended this result to $\mathbf{R}^4$.

In 2002, Corneli et al. [Corneli et al. 03] solved the double bubble problem for the flat two-torus, showing that there are five types of minimizers (one of which only occurs on the hexagonal torus). The proof relies on regularity, a variational component bound due to Wichiramala [Morgan and Wichiramala 02], and combinatorial and geometric classification.

1.3 Double Bubbles in $\mathbf{T}^3$

Even the single bubble for the three-torus is not yet completely understood. There are, however, partial results. The smallest enclosure of half of the volume of the torus was shown by Barthe and Maurey [Barthe and Maurey 00, Section 3] to be given by two parallel two-tori. Morgan and Johnson [Morgan and Johnson 00, Theorem 4.4] showed that the least-area enclosure of a small volume is a sphere. Spheres, tubes around geodesics, and pairs of parallel two-tori are shown to be the only types of area-minimizing enclosures for most three-tori by work of Ritoré and Ros ([Ritoré 97, Theorem 4.2], [Ritoré and Ros 96]).

1.4 Small Comparable Volumes

We prove Theorem 4.1 by showing that every sequence of area-minimizing double bubbles with decreasing volume and fixed volume ratio contains standard double bubbles. The main difficulty lies in bounding the curvature. Once this is accomplished, it is possible to show that the bubble lies inside a small ball that lifts to $\mathbf{R}^3$, where by [Hutchins et al. 02], a minimizer is known to be standard.

The main idea of the argument is as follows: From a given sequence of double bubbles with shrinking volumes and fixed volume ratio, we generate a new sequence by rescaling the ambient manifold at each stage so that one of the volumes is always equal to one. We can then apply compactness arguments and area estimates to show that certain subsequences of sequences obtained by rigid motions of the ambient manifold have nontrivial limits. These limits are used to obtain a curvature bound on a subsequence. With such a bound, we apply a monotonicity result to conclude that for small volumes, the double bubble is contained in a small ball. We conclude that it must be the same as the minimizer in $\mathbf{R}^3$ or $\mathbf{R}^4$, i.e., it must be the standard double bubble by [Hutchins et al. 02] or [Reichardt et al. 03]. The result extends to any flat three- or four-manifold with compact quotient by the isometry group.

1.5 Plan of the Paper

Section 2 reviews the methods that led to our Central Conjecture 2.1 and to Figures 1 and 2. Section 3 surveys some sub-conjectures. Section 4 is the statement and proof of the main theorem on small volumes. Section 5 shifts from the cubic torus to other tori and discusses other conjectures and candidates, including a "Hexagonal Honeycomb."

2. THE CONJECTURE

2.1 Generating Candidates

Participants in the CMI/MSRI Summer School proposed many possibilities for double bubbles in the three-torus $\mathbf{T}^3$in brainstorming sessions.

Standard Double Bubble. The least-area double bubble in $\mathbf{R}^3$.

Delaunay Chain. Two stacked "beads" or "drums" wrapping around one of the periods of the torus. The lateral surfaces are Delaunay surfaces (constant-mean-curvature surfaces of rotation).

Cylinder Lens. A cylinder wrapping around a period of the torus, with a small bubble attached.

Double Cylinder. Two cylinders wrapping around a period of the torus. Transverse sections by an orthogonal plane (i.e., a two-torus) are the standard double bubble in two dimensions. This is the first of a family of solutions we could describe as "cylinders over minimizing double bubbles in the two-torus."

Cylinder String. A cylinder over a "Symmetric Chain" in $\mathbf{T}^2$.

Slab Cylinder. Two parallel flat two-tori with a cylindrical bubble attached to one of them. This is a cylinder over a "Band Lens" in $\mathbf{T}^2$.

Double Slab. Three parallel flat two-tori (cylinder over a "Double Band" in $\mathbf{T}^2$).

Slab Lens. Two parallel flat two-tori, one of which has a small lens-shaped bubble stuck in it.

Center Bubble. Two close-to-parallel two-tori with a close-to-cylindrical bubble between them. The planes buckle slightly (as can be shown by elementary stability considerations) and the bubble stuck between them is not quite round (it bulges slightly in the direction of each of the four corners of the fundamental domain pictured in Figure 1).

SDB	=	Standard Double Bubble
DC	=	Delaunay Chain
CL	=	Cylinder Lens
CC	=	Cylinder Cross
2C	=	Double Cylinder
SL	=	Slab Lens
CB	=	Center Bubble
CS	=	Cylinder String
SC	=	Slab Cylinder
2S	=	Double Slab

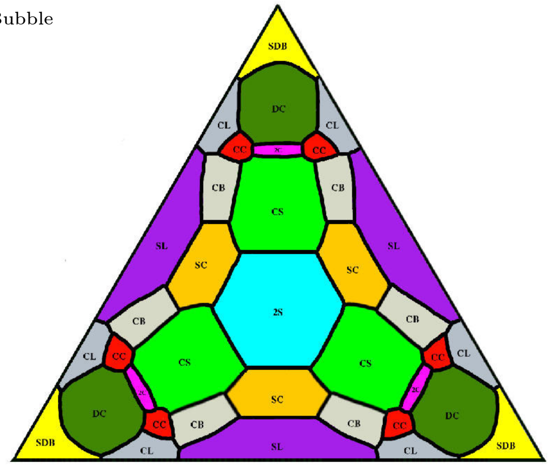

FIGURE 2. Phase portrait: volumes and corresponding double bubble. In the center both regions and the complement have one third of the total volume; along the edges, one volume is small; in the corners, two volumes are small.

Transverse Cylinders. Two cylinders wrapping around two different periods of the three-torus that touch each other to reduce surface area.

Hydrant Lens. Known as "Scary Gary" after its inventor, Gary Lawlor; this is essentially the Schwarz P surface with a small bubble attached.

Double Hydrant. Another fanciful proposal based on the Schwarz P surface.

Inner Tube. A cylinder that wraps around one of the periods of the three-torus with a toroidal bubble wrapped around it.

When preparing for the calculations that resulted in the phase diagram of Figure 2, we anticipated the possibility that a minimizing bubble topology might have been missed. In fact, two additional candidate topologies were found by systematic application of this heuristic:

> When the two volumes are very small, the standard double bubble should be optimal. As one or both of the two volumes grow, a bubble that collides with itself should open up to reduce perimeter, whereas if two different bubbles collide, they should stick.

This procedure incorporates the assumption that all three regions associated with a minimizer will be connected. The new candidates we found were as follows:

Cylinder Cross. A cylinder wrapping around one period of the torus with an attached bubble that wraps around one of the perpendicular directions. (Obtained from the Cylinder Lens when the small region grows.)

Center Cylinder. Two parallel flat two-tori with a cylinder going through both of them and wrapping around a period of the three-torus. (Obtained from the Center Bubble when the small region grows or from the Transverse Cylinders as one of the cylinders grows.)

2.2 Producing the Phase Diagram

Brakke's *Surface Evolver* [Brakke xx] was used to closely approximate the minimal area that a double bubble of each type needs to enclose specified volumes. Our initial simulations were done with volume increments of 0.05 (from a total volume of 1) for the initial set of brainstormed candidates. The results of these calculations were sufficient to convince us that the Hydrant Lens and the Double Hydrant were inefficient and that the Inner Tube was in fact unstable (which is why we could not draw a picture of it with *Surface Evolver* for inclusion in this paper). A rough version of Figure 2 was then obtained using a volume increment of 0.01. From the data obtained in these computations, we found that each of the double bubbles pictured in Figure 1 was the least-area competitor for some volume triple $v_1 : v_2 : v_3$. The phase diagram appearing in Figure 2 is the result of refining

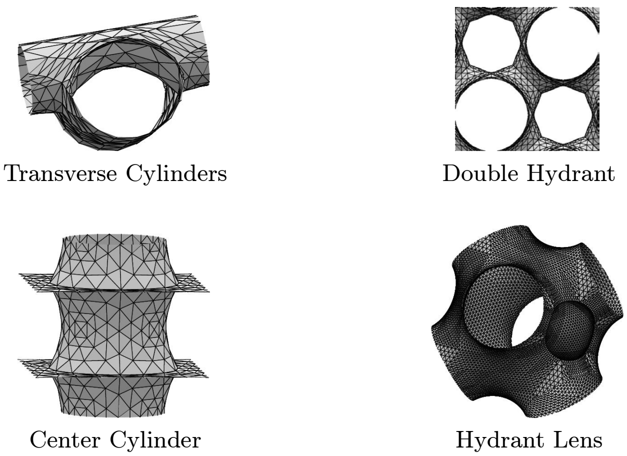

FIGURE 3. Inefficient double bubbles.

our second series of computations along the boundaries between the regions using a volume increment of 0.005.

Conjecture 2.1. (Central Conjecture.) *The ten double bubbles pictured in Figure 1 represent each type of surface area-minimizing two-volume enclosure in a flat, cubic three-torus, and these types are minimizing for the volumes illustrated in Figure 2.*

2.3 Comments

One might guess that minimizers would be formed from topological spheres and tori that go around at least one period of $\mathbf{T}^3$. This was borne out in our computations. It is interesting to note that although the Transverse Cylinders pictured in Figure 3 match this description, they were never seen to be minimizing.

A challenging problem related to the problem of area minimization is that of finding all of the stable double bubbles in $\mathbf{T}^3$; there are probably quite a few that we have not looked at. Note that a given type from Figure 1 might be stable for a much wider range of volumes than those for which it actually minimizes surface area.

It is worth observing that all of the conjectured minimizers for the double bubble problem on $\mathbf{T}^2$ are echoed here in at least two ways: several conjectured minimizers are $\mathbf{T}^2$ minimizers $\times$ $\mathbf{S}^1$, and others are more direct analogues, for example, the Delaunay and Symmetric Chains. See Corneli et al. [Corneli et al. 03] for more on the $\mathbf{T}^2$ minimizers.

A few words about the accuracy of the simulations are in order. The commands used in our final stage of numerics were as follows:

```
// five cycles of refining the mesh and
//  decreasing area
{{u; g; u} 10; {u; r; u}} 5;
// decrease area ten more times at the end
{u; g; u} 10;
```

These commands produced pictorial output that perfectly matches our idea of what each conjectured minimizer should look like.

3. SUBCONJECTURES

We now present a list of natural subconjectures about bubbles in the flat cubic three-torus $\mathbf{T}^3$ that are suggested either by the phase diagram or by examination of the pictures made by *Surface Evolver*.

One immediate observation is that the edges of our phase diagram appear to characterize single bubbles in $\mathbf{T}^3$.

Conjecture 3.1. (Ritoré and Ros [Ritoré 97], [Ritoré and Ros 96], [Ros 01].) *The optima for the isoperimetric problem in $\mathbf{T}^3$ are the sphere, cylinder, and slab.*

The following two conjectures are perhaps the most intuitive things we would expect to be true about bubbles in $\mathbf{T}^3$:

Conjecture 3.2. *An area-minimizing double bubble in $\mathbf{T}^3$ has connected regions and complement.*

Conjecture 3.3. *There is an $\epsilon > 0$ such that if v_1, v_2 are both less than ϵ, the standard double bubble of volumes v_1 and v_2 is optimal in $\mathbf{T}^3$.*

In Theorem 4.1, we prove that there is such an ϵ for every fixed ratio of volumes.

If it were known for some reason that the number of components of a minimizer was relatively small, then we think the next two conjectures might admit fairly easy proofs:

Conjecture 3.4. *For one very small volume and two moderate volumes, the Slab Lens is optimal.*

Conjecture 3.5. *The first phase transition as small equal or close to equal volumes grow is from the standard double bubble to a chain of two bubbles bounded by Delaunay surfaces.*

Delaunay surfaces are constant-mean-curvature surfaces of rotation, and as such contain an $\mathbf{S}^1$ in their symmetry group. From the *Surface Evolver* pictures, it appears that the conjectured surface area minimizers always have the maximal symmetry, given the constraints, a fact that leads to the following natural conjecture.

Conjecture 3.6. *All isotopies that are not isometries decrease a minimizer's symmetry group.*

We will conclude the section with a proposition that establishes a very limited symmetry property for all double bubbles in the two torus (including nonminimizing bubbles). Specifically, we prove that for any double bubble, there is a pair of parallel two-tori that cut both regions in half. Hutchings [Hutchings 97, Theorem 2.6] was able to show that in $\mathbf{R}^n$ a minimizing double bubble has two perpendicular planes that divide both regions in half. He used this result in the proof that the function that gives the least-area to enclose two given volumes in $\mathbf{R}^n$ is concave. We conjecture that a similar concavity result also holds for the three-torus:

Conjecture 3.7. *The least area to enclose and separate two given volumes in the three-torus is a concave function of the volumes.*

If one could prove Conjecture 3.7, one would then be able to apply other ideas in Hutchings' paper [Hutchings 97, Section 4] to obtain a functional bound on the number of components of a minimizing bubble.

Proposition 3.8. (Deluxe Ham Sandwich Theorem.)* *If a double bubble lies inside a solid two-torus inside a rectangular three-torus, $D^2 \times \mathbf{S}^1 \subset \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, then there is a pair of parallel two-tori that divide both volumes in half.*

Proof: This is a generalization of the standard argument for the ham sandwich theorem in Euclidean space. We will regard the three-torus as the result of identifying the faces of a rectangular cube $[-n, n] \times [-m, m] \times [0, p]$ in $\mathbf{R}^3$. We may assume that the solid torus is a vertical cylinder with base centered at the origin in the $x - y$ plane. Now we rotate this solid cylinder, and index the rotation by $\alpha \in [0, 2\pi]$. Then for each $\alpha \in [0, \pi]$, rotate the cylinder (along with the surface that lies inside) by α in the $x - y$ plane and consider the family of pairs of planes $\{y = c, y = c + m\}$, where $c \in [-m, 0]$ For each $\alpha \in [0, \pi]$, there is at least one such pair of planes that cuts the volume V_1 in half, in the sense that half of V_1 is in the inside of the planes, and half of V_1 is on the outside of the two planes. There may be an interval of such pairs for a given α. However, some one parameter family of these planes may be chosen which varies continuously as a function of α, and coincides at 0 and π. Since by the time α reaches π the two portions of V_2 have switched sides, there must be some pair of planes that divides V_2 in half. The restriction of these planes to the domain $[-n, n] \times [-m, m] \times [0, p]$ glue up to a pair of two-tori in the three-torus that divides both regions in half. □

4. SMALL COMPARABLE VOLUMES

Conjecture 3.3 asserts that two small volumes in $\mathbf{T}^3$ are best enclosed by a standard double bubble. In this section, we will prove Theorem 4.1, which says that for any fixed volume ratio, the standard double becomes optimal when the volumes are sufficiently small. This result holds for a relatively broad class of three- and four-dimensional manifolds (and see Remark 4.4).

The standard double bubble consists of three spherical caps meeting at 120 degrees (see Figure 2). If the volumes are equal, the middle surface is planar. The standard double bubble is known to be minimizing for all volume pairs in $\mathbf{R}^3$ [Hutchins et al. 02] and $\mathbf{R}^4$ [Reichardt et al. 03].

Theorem 4.1. *Let M be a flat Riemannian manifold of dimension three or four such that M has compact quotient by its isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$, such that if $0 < v < \epsilon$, an area-minimizing double bubble in M of volumes $v, \lambda v$ is standard.*

*Name suggested by Eric Schoenfeld, who gave an independent proof for $\mathbf{T}^2$.

Remark 4.2. Our assumption on the isometry group guarantees solutions to the double bubble problem for all volume pairs (the proof is the same as in Morgan [Morgan 00, Section 13.7]). Not every flat 3-manifold has compact quotient by its isometry group. Euclidean 3-space modulo a glide reflection or a glide rotation is an example.

It is helpful to have the overall strategy of the proof in mind before we go on. Our goal is to prove that a minimizing double bubble of volumes v, λv lies inside a trivial ball in M when v is small enough. The results for Euclidean space then prove our claim.

For the proof, we will regard a double bubble as a pair of three- or four-dimensional rectifiable currents, R_1 and R_2, each of multiplicity one, of volumes $V_1 = \mathbf{M}(R_1)$ and $V_2 = \mathbf{M}(R_2)$. The total area of such a double bubble is $\frac{1}{2}(\mathbf{M}(\partial R_1) + \mathbf{M}(\partial R_2) + \mathbf{M}(\partial(R_1 + R_2)))$. Here $\mathbf{M}$ denotes the mass of the current, which can be thought of as the Hausdorff measure of the associated rectifiable set (counting multiplicities). For a review of the pertinent definitions, see the notes from Morgan's course at the CMI/MSRI Summer School [Morgan and Ritoré 01] or the texts by Morgan [Morgan 00] or Federer [Federer 69]. By the Nash and subsequent isometric embedding theorems, we may assume M is a submanifold of some fixed $\mathbf{R}^N$.

We will consider a sequence of area-minimizing double bubbles in M enclosing the volumes v and λv, as $v \to 0$. If the theorem were false, there would be some such sequence with no standard double bubbles in it. We prove that every sequence of fixed volume ratio double bubbles with shrinking volumes contains standard double bubbles.

For each v, M_v will denote $s_v(M)$ in $\mathbf{R}^N$, where s_v is the scaling map that takes regions with volume v to similar regions with volume 1. In particular, s_v maps our area-minimizing double bubble enclosing volumes v and λv to a double bubble which we call S_v that encloses volumes 1 and λ, and is of course area-minimizing for these volumes.

The first step is to show that once M_v has been suitably translated and rotated, the sequence S_v has a subsequence that converges as $v \to 0$ with $V_1 \neq 0$ in the limit. Our argument also shows that there is a differently modified subsequence that converges to a limit with $V_2 \neq 0$, but does not show that there is a subsequence where both volumes are nonzero in the limit. This is because while we are exerting ourselves trapping the first volume in a ball, the second one may wander off to infinity.

The second step in the proof is to use these limits to obtain a weak bound on the curvature of a subsequence of S_v. We need this in order to apply the monotonicity theorem for mass ratio [Allard 72, Section 5.1(1)], which says that a small ball around any point on a surface with weakly bounded mean curvature contains some substantial amount of area.

The third step is to show that all of the surface area of each element of our subsequence is contained in some ball in M_v that has fixed radius for all v. Eventually, as v shrinks and M_v grows, this ball will have to be trivial in M_v.

We then use the result that the optimal double bubble in $\mathbf{R}^3$ is standard to show that our subsequence has a tail comprised of standard double bubbles. The desired conclusion then follows from the fact that S_v is simply a scaled version of the original double bubble containing volumes v and λv. Thus there is no sequence of shrinking fixed-ratio area-minimizing double bubbles in M that does not contain standard double bubbles.

We focus on the case of dimension three; the proof for dimension four is essentially identical.

The following lemma is needed in the first step of the proof.

Lemma 4.3. *There is a $\gamma > 0$ such that if R is a region in an open Euclidean 3-cube K and* $\mathrm{vol}(R) \leq \mathrm{vol}(K)/2$, *then*

$$\mathrm{area}(\partial R) \geq \gamma(\mathrm{vol}(R))^{2/3}.$$

Proof: Let γ_0 be such an isoperimetric constant for a cubic three-torus, so that $\mathrm{area}(\partial P) \geq \gamma_0(\mathrm{vol}(P))^{2/3}$ for all regions $P \subseteq \mathbf{T}^3$. Such a γ_0 exists by the isoperimetric inequality for compact manifolds [Morgan 00, Section 12.3]. Make the necessary reflections and identifications of the cube K to obtain a torus containing a region R' with eight times the volume and eight times the surface area of R. The claim follows, with $\gamma = \gamma_0/2$. □

Proof of Theorem 4.1:

Step 1. The sequence $S_v \subset M_v$, where each M_v has been suitably translated and rotated, has a subsequence that converges with $V_1 \neq 0$ in the limit.

We first show the existence of a covering $\mathcal{K}_v$ of M_v with bounded multiplicity, consisting of 3-cubes contained in M_v, each of side-length L, for any $L > 0$. Lemma 4.3 will give us a positive lower bound on the volume of the part of R_1 that is inside one of these cubes for each S_v. We will then apply a standard compactness theorem to show that a subsequence of the sequence of S_v converges.

Take a maximal packing of M_v by balls of radius $\frac{1}{4}L$. Enlargements of radius $\frac{1}{2}L$ cover M_v. Circumscribed 3-cubes in M_v of edge-length L provide the desired covering $\mathcal{K}_v$. To see that the multiplicity of this covering is bounded, consider a point $p \in M_v$. The ball centered at p with radius $2L$ contains all the cubes that might cover it, and the number of balls of radius $\frac{1}{4}L$ that can pack into this ball is bounded, implying that the multiplicity of $\mathcal{K}_v$ is also bounded by some $m > 0$.

Now let $\mathcal{K}_v$ be a covering as above, with $L = 2$. By Lemma 4.3, there is an isoperimetric constant γ such that

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma(\text{vol}(R_1 \cap K_i))^{2/3},$$

and therefore, since $\max_k \text{vol}(R_1 \cap K_k) \geq \text{vol}(R_1 \cap K_i)$ for any i,

$$\text{area}(\partial(R_1 \cap K_i)) \geq \gamma \frac{\text{vol}(R_1 \cap K_i)}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}. \tag{4–1}$$

Note that the total area of the surface is greater than $1/m$ times the sum of the areas in each cube, and the total volume enclosed is less than the sum of the volumes, so summing Equation 4–1 over all the cubes K_i in the covering $\mathcal{K}_v$ yields

$$\text{area}(S_v) \geq \text{area}(\partial R_1) \geq m\gamma \frac{V_1}{(\max_k \text{vol}(R_1 \cap K_k))^{1/3}}$$

and

$$(\max_k(\text{vol}(R_1 \cap K_k)))^{1/3} \geq m\gamma \frac{V_1}{\text{area}(S_v)} \geq \delta,$$

for some $\delta > 0$, because $V_1 = 1$ and $\text{area}(S_v)$ is bounded (there is a bounded way of enclosing the volumes, and S_v has the same amount of area or less).

Translate each M_v so that a cube K_k that maximizes $\text{vol}(R_1 \cap K_k)$ is centered at the origin of $\mathbf{R}^N$, and rotate so that the tangent space of each M_v at the origin is equal to a fixed $\mathbf{R}^3$ in $\mathbf{R}^N$. The limit of the M_v will be equal to this $\mathbf{R}^3$. Since a cube with edge-length L centered at the origin fits inside a ball of radius $2L$ centered at the origin, we have

$$\text{vol}(R_1 \cap B(0, 2L)) \geq \delta^3$$

for every S_v. By the compactness theorem for locally integral currents ([Morgan 00, pp. 64, 88], [Simon 84, Section 27.3, 31.2, 31.3]), we know that a subsequence of the S_v has a limit, which we will call D, with the property that $\text{vol}(R_1) \geq \delta^3$. This completes the first step.

Since D is contained in the limit of the M_v, namely, the copy of $\mathbf{R}^3$ chosen above, and each S_v is minimizing for its volumes, a standard argument shows that the limit D is the area-minimizing way to enclose and separate the given volumes $\text{vol}(R_1) \geq \delta^3$ and $\text{vol}(R_2)$ in $\mathbf{R}^3$ (see [Morgan 00, 13.7]). In the limit, $\text{vol}(R_2)$ could be zero, in which case D is a round sphere. If both volumes are nonzero, D is the standard double bubble ([Hutchins et al. 02] and [Reichardt et al. 03], or see [Morgan 00, Chapter 14]).

Step 2. By taking a subsequence if necessary, we may assume there is a weak bound on the curvature of the S_v.

To show that the mean curvature is weakly bounded is to show that there is a C such that for any S_v, and any direction in the two-dimensional space of possible volume changes, $|\frac{dA}{dV}| \leq C$ for smooth variations. Thus it suffices to produce smooth variations such that changes in the volume of S_v and in the area of ∂S_v are controlled, that is, that the magnitude of the change in volume is bounded from below and the magnitude of the change in area is bounded from above.

We first show that the magnitude of the change in volume is bounded below. Take a smooth variation vector field F in $\mathbf{R}^n$ such that for D,

$$dV_1/dt = \int_{\partial R1} (F \cdot n)\ dA = c \neq 0$$

and

$$dV_2/dt = 0.$$

(Note that we need the first volume to be nonzero, or its variation could be zero.) For v small enough, the subsequence of S_v associated with the limit D has the property that dV_1/dt is approximately equal to the constant c and dV_2/dt is approximately equal to 0.

By the argument in the first step, we can translate and rotate each M_v so that, after taking a subsequence again if necessary, S_v converges to a minimizer D' in $\mathbf{R}^3$ where the second volume is nontrivial. This time we take a smooth variation vector field F', such that the images of S_v under the appropriate rigid motions have the property that dV_1/dt is approximately equal to 0 and dV_2/dt is approximately equal to some constant $c' \neq 0$, for v small enough.

Note that the change in volume is independent of rigid motions; in particular, we can translate both F and F' back along the rigid motions to act on S_v. This proves that for this sequence, the magnitude of the change in volume is bounded below.

Now we need to show that the change in area is bounded above. This follows from the fact that every rectifiable set can be thought of as a varifold [Morgan 00, Section 11.2]. By compactness for varifolds [Allard 72, Section 6], the S_v, situated so that the first volume does not disappear, converge as varifolds to some varifold, J. The first variations of the varifolds also converge, i.e., $\delta S_v \to \delta J$; see [Allard 72]. The first variation of a varifold is a function representing the change in area. Therefore, far enough out in the sequence the change in area of the S_v under F is bounded close to the change in area of J under F, which is finite. Similarly, the change in area of the S_v under F' is bounded above.

We conclude that, after restricting to a subsequence, S_v has the property that $|\frac{dA}{dV}|$ is bounded for two independent directions in the two-dimensional space of volume changes, and hence for the entire space. This completes the second step.

Step 3. When v is small, all of the elements of a subsequence of $S_v \subset M_v$ are contained in a ball with a fixed radius that can be lifted to $\mathbf{R}^3$.

As v shrinks, the area of each element of the subsequence S_v found in Step 2 is bounded above by A, since our double bubbles are area-minimizing, and a nearly Euclidean double bubble is one candidate. By monotonicity of mass ratio [Allard 72, Section 5.1(1)], every unit-radius ball centered at a point of S_v contains at least area ϵ, for some $\epsilon > 0$. Therefore, there are at most A/ϵ such disjoint balls.

We claim that the S_v are eventually connected, which will imply that the diameter S_v is bounded above by $2A/\epsilon$. Indeed, unless the S_v are eventually connected, one can arrange to get in the limit a disconnected minimizer in $\mathbf{R}^3$ by translating and centering nontrivial volume on two different points. This is false, so we conclude that S_v is contained in a ball of radius $2A/\epsilon$ for all small v.

Since our original manifold has compact quotient by its isometry group, we may quotient by isometries to get a compact 3-manifold. This implies that balls of a certain radius or smaller are topologically trivial. Hence, as we expand the manifold, balls of radius $2A/\epsilon$ can be lifted to $\mathbf{R}^3$, which means that they are Euclidean. This completes the third step.

Since S_v is eventually contained in a Euclidean ball, the tail of this sequence is made up entirely of standard double bubbles ([Hutchins et al. 02], [Morgan 00, Chapter 14]). Recall that our sequence S_v is simply a scaled version of the original sequence of minimizing double bubbles in M enclosing volumes v and λv, where we have restricted to subsequences as necessary. Therefore, any sequence of area-minimizing double bubbles with shrinking volumes and fixed volume ratio contains standard double bubbles. We conclude that the minimizing double bubble with volume ratio λ is standard when v is small enough. $\square$

Remark 4.4. Given n and m, essentially the same argument shows that for any smooth n-dimensional Riemannian manifold with compact quotient by the isometry group, given $0 < \lambda \leq 1$, there are C, $\epsilon > 0$, such that for any $0 < v < \epsilon$, a minimizing cluster with m prescribed volumes between λv and v lies inside a trivial ball. For the case of flat three- or four-dimensional manifolds with compact quotient by the isometry group, this means that small clusters of bubbles with comparable volumes are the same as the Euclidean minimizer containing the same volumes. To deduce the corresponding result in the nonflat case would require knowing that convergence weakly and in measure, under bounded mean curvature, implies C^1 convergence, as is known for hypersurfaces without singularities ([Allard 72, Section 8], see [Morgan 01, Section 1.2]).

5. SPECIAL TORI

Changing the shape of the torus, by stretching it or by skewing some or all of its angles, would certainly change the phase diagram of Figure 2.

Conjecture 5.1. *In the special case of a very long three-torus, i.e., with side lengths $\{\frac{1}{\epsilon^2}, \epsilon, \epsilon\}$ $(\epsilon \to 0)$, the Double Slab is optimal for most volumes.*

Special tori may have special minimizers:

Conjecture 5.2. *For the special case of a torus based on a short hexagonal prism, the Hexagonal Honeycomb prism of Figure 4 is an area-minimizing double bubble when both regions and the exterior all have equal or close to equal volumes.*

Indeed, for such volumes, the Hexagonal Honeycomb ties the Double Slab, just as in the hexagonal two-torus, a Hexagonal Tiling ties the Double Band [Corneli et al. 03].

For multiple enclosed regions, we would expect to find minimizers that lift to $\mathbf{R}^3$ as periodic foams with cells of finite volume, such as Kelvin's foam or the Weaire-Phelan foam [Kusner and Sullivan 96].

However, there is no minimizing *double* bubble in any torus that lifts to a division of $\mathbf{R}^3$ into bounded regions. Indeed, as pointed out by Adams, Morgan, and Sullivan

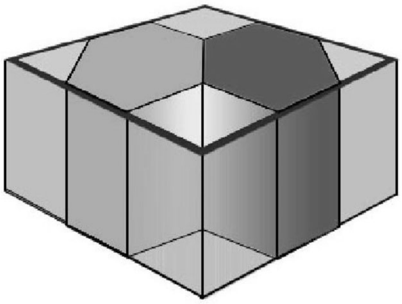

FIGURE 4. Hexagonal honeycomb

[Adams et al. 2003], by Lebesgue's Covering Theorem [Hurewicz and Wallman 41, Theorem IV2], such regions must sometimes meet in fours or more.

This suggests that it is likely that there are no other special minimizers for the double bubble problem.

Conjecture 5.3. *The double bubbles of Figure 1 together with the Hexagonal Honeycomb of Figure 4 comprise the complete set of area-minimizing double bubbles for all three-tori.*

In light of the fact that the triple bubble problem in the torus seems likely to produce many interesting candidates, we conclude with one final conjecture.

Conjecture 5.4. *For the triple bubble problem in a flat cubic three-torus in the case where one of the volumes is small, the minimizers will look like the double bubbles of Figure 2 with a small ball attached. The phase diagram will look just like our Figure 3.*

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